Population Diversity of Particle Swarms

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Abstract— **In the field of evolutionary computation, an important attribute of a population is diversity. This paper proposes a method for measuring the diversity of a particle swarm optimization population. It involves the measurement of position and velocity attributes of the particles that comprise the population. The proposed method is computationally straightforward and is adaptable to other evolutionary algorithms.**

I. Introduction

A. Background

PARTICLE swarm optimization (PSO) is an evolutionary computation technique developed by Kennedy and Eberhart in 1995 [1]-[3]. It is being researched and utilized in over 30 countries.

The process for implementing the *global* version of PSO is as follows:

1) Initialize a population (array) of particles with random positions and velocities on *n* dimensions in the problem space.
2) For each particle, evaluate the desired optimization fitness function in *n* variables.
3) Compare particle's fitness evaluation with particle's *pbest*. If current value is better than *pbest*, then set *pbest* value equal to the current value, and the *pbest* location equal to the current location in *n*--dimensional space.
4) Compare fitness evaluation with the population's overall previous best. If current value is better than *gbest*, then reset *gbest* to the current particle's array index and value.
5) Change the velocity and position of the particle according to (1) and (2), respectively [4]–[6]:

$$v_{id} = w * v_{id} + c_1 * rand() * (p_{id} - x_{id}) + c_2 * Rand() * (p_{gd} - x_{id}) \quad (1)$$

$$x_{id} = x_{id} + v_{id} \quad (2)$$

After step 5, loop to step 2 until a criterion is met, usually a sufficiently good fitness or a maximum number of iterations (generations).

Particle swarm optimization is especially useful for obtaining answers to problems involving multiple objectives and multiple constraints. It has outperformed other algorithms in a number of benchmark tests, and a number of researchers have developed methodologies for its utilization for these types of problems.

There is also a *local* version of PSO in which, in addition to *pbest*, each particle keeps track of the best solution, called *lbest*, attained within a *local* topological neighborhood of particles.

The positions and velocities of a population of particles can be represented in vector format as follows, where *m* is the population size of the swarm, and *n* is the number of dimensions (variables) for each particle.

$$x_i = \{x_{i1}, x_{i2}, \ldots, x_{in}\}, \ i = 1, \ldots, m \quad (3)$$

$$v_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}, \ i = 1, \ldots, m \quad (4)$$

The population of particles' positions and velocities can be represented in matrix form as follows:

$$X = (x_i, x_2, \cdots, x_m)^T = \begin{pmatrix} x_{11}x_{12}\cdots x_{1n} \\ x_{21}x_{22}\cdots x_{2n} \\ \vdots \quad \ddots \\ x_{m1}x_{m2}\cdots x_{mn} \end{pmatrix} \quad (5)$$

$$V = (v_i, v_2, \cdots, v_m)^T = \begin{pmatrix} v_{11}v_{12}\cdots v_{1n} \\ v_{21}v_{22}\cdots v_{2n} \\ \vdots \quad \ddots \\ v_{m1}v_{m2}\cdots v_{mn} \end{pmatrix} \quad (6)$$

B. Fitness-based Population Diversity

In the literature, including a paper co-authored by the authors of this paper [7], the diversity of a genetic algorithm (GA) population has been calculated using the standard deviation of the individual fitness values of population members. This is, however, an indirect metric for population diversity; fitnesses are attributes of population behavior (phenotype), rather than direct diversity measures at the information-theoretic level. The diversity of a GA population has also been calculated as the average Euclidian distance from the GA population's average location vector [8].

The authors propose to view diversity from more of an information-theoretic perspective. Such an approach should be based on something more closely related to entropy. For a particle swarm, the positions and velocities of the particles provide such a basis. In the following sections, we present basic metrics for population position diversity and population velocity diversity, and then discuss ways to combine them

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into a unified diversity metric for particle swarm diversity. With minor modifications, this approach is applicable to other evolutionary algorithms.

II. Position Based Population Diversity

A. Element-wise Population Position Diversity

Based on previous work done on GA population diversity [8],[9], the elements of all dimensions in an individual can be equally weighted. (It should be noted that in the two cited references, each element is a binary gene, having a value of either 0 or 1, rather than a real value.)

The superscript index *p* in D^p (Equation 8) stands for the population position diversity with respect to the particle's position x_i. In subsequent sections, the superscript *v* stands for population velocity diversity.

$$\bar{x} = \frac{1}{n \times m} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \tag{7}$$

$$D^p = \frac{1}{n \times m} \sum_{i=1}^{m} \sum_{j=1}^{n} [x_{ij} - \bar{x}]^2 \tag{8}$$

B. Euclidean Distance-based Population Diversity

In this method, the Euclidean distance is measured between pairs of population members for all possible combinations.

$$d^p(x_i, x_j) = \left\| x_i - x_j \right\| \tag{9a}$$

$$\tilde{d}^p(x_i, x_j) = \frac{d^p(x_i, x_j)}{\left\| a - b \right\|} \tag{9b}$$

$$D_{ED}{}^p = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \tilde{d}(x_i, x_j) \tag{10}$$

In this measure, [a, b] is the dynamic range of the particle's position, that is, the particles are limited to "fly" within the range [a, b]. This type of diversity metric was discussed in Wen *et al.* [10] and Wen *et al.* [11].

C. Dimension-wise Population Position Diversity

For a particle swarm, instead of considering whether the particles are close to each other by looking at a single distance measure separating them, we measure the diversity of each dimension's positions by considering the position values on each dimension of the particles in the population. In the equations below, *m* is the population size, and the subscript *i* is a dimension which varies from 1 to *m*. Also, *n* is the number of dimensions (variables) in each particle, and the subscript *j* is an index that varies from 1 to *n*.

$$\bar{x}_j = \frac{1}{m} \sum_{i=1}^{m} x_{ij}$$

$$D_j^p = \frac{1}{m} \sum_{i=1}^{m} [x_{ij} - \bar{x}_j]^2 \qquad \text{(11a, 11b)}$$

Therefore, we have a dimension-wise position diversity vector $(D_1^p, D_2^p, \cdots, D_n^p)$. Based on this *n*-dimensional diversity measure, there are several ways to measure the swarm population position diversity.

1) Weighted Summation Position Diversity

In the first position diversity measure we examine, the individual dimensions (parameters) are assigned different weights, resulting in the weighted summation position diversity over all dimensions as defined in (12).

$$D_{WS}^p = \sum_{j=1}^{n} w_j D_j{}^p \tag{12}$$

Each w_j is a positive value (weight) less than or equal to 1. If all the dimensions are treated equally, we have $w_j = 1/n$ for all values of *j*. Then (12) can be revised as (13) for weighted summations with equal weights.

$$D_{WSew}^p = \frac{1}{n} \sum_{j=1}^{n} D_j{}^p \tag{13}$$

2) Weighted Maximization Position Diversity

Another measure that might be of interest is a weighted maximum position diversity, which is the weighted maximum value calculated on any individual dimension, as indicated in (14).

$$D_{WM}^p = \max\{w_j D_j{}^p\} \qquad j = 1, \cdots, n \tag{14}$$

Again, w_j is the weight for the D_j^p position diversity for one dimension defined in (11b).

3) Position Diversity Vector Length

It may be useful to calculate the Euclidean length of the position diversity vector, as illustrated in (15).

$$D_L^p = \sqrt{\sum_{j=1}^{n} D_j^{p\,2}} \tag{15}$$

4) Normalized Dimension-wise Population Position Diversities

Another method to calculate the position diversity values analogous to (12)–(15) is to normalize the position values before calculating the position diversity.

In most implementations of PSO, the particles are restricted to an initial range of values (dynamic range) for each parameter (dimension). Assuming the particle is initially limited to the dynamic range [a_j, b_j] on dimension *j*, then the normalized particle position value is given in (16).

$$\hat{x}_{ij} = \frac{x_{ij}}{| b_j - a_j |} \qquad a_j \le x_{ij} \le b_j; \qquad i = 1, \cdots, m; \qquad j = 1, \cdots n \tag{16}$$

The normalized dimension-wise position diversity can then be calculated as in (17a) and (17b).

$$\bar{x}_j = \frac{1}{m} \sum_{i=1}^{m} \hat{x}_{ij}$$
$$D_{jN}^{p} = \frac{1}{m} \sum_{i=1}^{m} [\hat{x}_{ij} - \bar{x}_j]^2 \tag{17a, 17b}$$

The population position normalized diversity can then be obtained by using the weighted summation, weighted maximization, or diversity vector length approach defined in (12)–(15).

III. Population Velocity Diversity

A feature of PSO not found in other evolutionary algorithms is that there is a velocity, in addition to a position, associated with each population member (particle). Therefore, we should not only consider the population diversity from the perspective of position, but also from the perspective of velocity.

To distinguish between position and velocity, we use the superscript index *p* to represent diversity with regards to the particles' positions, and the superscript index *v* to represent diversity with regards to the particles' velocities.

Velocity has two components: speed and direction. We consider particles' velocity from two perspectives: dimension by dimension, where we calculate speed diversity; and particle by particle, for which we calculate direction diversity. Since particles move along dimensions, it is logical to use intra-dimensional movement to calculate speed diversity. However, there is no intra-dimensional direction diversity. (See (1) and (2), from which it can be seen that calculations are done on a dimension-by-dimension basis.) We calculate direction diversity using the difference in angle between each particle's overall velocity vector and the average velocity vector for the swarm population.

A. Dimensional Speed Diversity

To calculate the dimensional speed diversity, each particle's speed is first averaged over all population members along a single dimension *j*, then, we calculate dimension-wise speed diversity, as shown in (18) and (19), where $D_j^{v\_ds}$ is the speed diversity on dimension *j*.

$$\bar{v}_j = \frac{1}{m} \sum_{i=1}^{m} v_{ij} \tag{18}$$

$$D_j^{v\_ds} = \frac{1}{m} \sum_{i=1}^{m} [v_{ij} - \bar{v}_j]^2 \tag{19}$$

B. Weighted Summation Dimensional Speed Diversity

In a manner similar to (12), we can calculate the weighted summation dimensional speed diversity over all dimensions.

$$D_{WS}^{v\_ds} = \sum_{j=1}^{n} w_j D_j^{v\_ds} \tag{20}$$

As before, we define w_i as a positive value less than or equal to 1. If all the dimensions are treated the same, we obtain $w_i = 1/n$. Then (20) can be written as (21).

$$D_{WSew}^{v\_ds} = \frac{1}{n} \sum_{j=1}^{n} D_j^{v\_ds} \tag{21}$$

C. Weighted Maximization Dimensional Speed Diversity

Analogous to the position diversity calculation of (14), we can calculate the weighted maximum dimensional speed diversity, shown in (22).

$$D_{WM}^{v\_ds} = \max\{w_j D_j^{v\_ds}\} \qquad j = 1, \cdots, n \tag{22}$$

Each w_j is the weight for $D_j^{v\_ds}$ as defined in the weighted summation calculation of (20).

D. Dimensional Speed Diversity Vector Length

It may be useful to calculate the Euclidean length of the dimensional speed diversity vector, as illustrated in (23).

$$D_L^{v\_ds} = \sqrt{\sum_{j=1}^{n} {D_i^{v\_ds}}^2} \qquad i = 1, \cdots, n \tag{23}$$

E. Particle Direction Diversity

To calculate the particle direction diversity, each particle's velocity elements are normalized using the unit length over all dimensions for that particle as shown in (24).

$$v_{ij}^{nor\_par} = \frac{v_{ij}}{\sqrt{\sum_{j=1}^{n} {v_{ij}}^2}} \tag{24}$$

We now calculate the normalized average velocity vector for the population of particles. We first average the velocity elements on each dimension, as shown in (25). This gives us the dimension-by-dimension elements of the average velocity vector.

$$v_j^{\dim\_avg} = \frac{1}{m}\sum_{i=1}^{m} v_{ij} \quad (25)$$

We then calculate the normalized average velocity vector elements (dimension-by-dimension) as shown in (26).

$$v_j^{nor\_avg} = \frac{v_j^{\dim\_avg}}{\sqrt{\sum_{j=1}^{n} {v_j^{\dim\_avg}}^2}} \quad (26)$$

We can now calculate the cosine of the angle θ between each normalized particle velocity vector and the normalized average particle velocity vector as in (27). The particle population direction diversity is then defined in (28).

$$\cos(\theta_i) = \sum_{j=1}^{n}\left(v_{ij}^{nor-par} \bullet v_j^{avg\_nor}\right) \quad (27)$$

$$D^{v\_dir} = \frac{1}{m}\sum_{i=1}^{m} \theta_i^2 \quad (28)$$

IV. Discussion

We should consider the position, speed, and direction diversities together, not only one or two of them at a time. There is a trade-off between position based population and velocity based population diversities. A way to calculate the overall diversity is shown in (29).

$$D = w_p D^p + w_{v\_ds} D^{v-ds} + w_{v\_dir} D^{v-dir} \quad (29)$$

The quantity D^{v-ds} can be any of the speed diversity metrics such as D_L^{v-ds} of (23). Without a loss of generality, the velocity, speed and direction diversities can be treated equally, that is, the weights w_p, w_{v_ds} and w_{v_dir} can be set to be equal. Other values are possible and will depend on the problem to be solved. The weights can also be a function of PSO performance and the diversities themselves. They can also be adjusted or evolved to reflect dynamic changes during the PSO search process.

In [12], Cui *et al.* describe a method to maintain diversity by degrading the reproduction probability of an individual in a GA based on the similarity between the individual and the remainder of the population. Exponential fitness rescaling is used based on the similarity $A_{i,j}$, which is a function of the average entropy of pairs of population members (30), where $H_{i,j}(2)$ is the average entropy of individuals i and j.

$$A_{i,j} = 1/\left(H_{i,j}(2)\right) \quad (30)$$

Although this method is not applicable to PSO since PSO doesn't use binary representation and it is not practical to calculate probabilities for each state of the swarm, the general approach of using entropy optimization principles is one that the authors are continuing to study. The three diversity measures proposed in this paper are possible sources of entropy-like metrics.

Another possible approach that we considered is pair-wise calculations of diversities (position, speed, and/or direction) similar in concept to the calculation of (10) for distance-based population diversity. However, the approaches we have used, such as (28), are significantly simpler and don't become computationally intensive for realistic problems. We acknowledge that some users might prefer, for particular reasons, to make pair-wise calculations for all three diversity measures, but we believe that the methods we propose will prove more effective for most applications.

Other metrics are enabled by the three diversity measures we propose. For example, the standard deviations of the diversity measurements may provide insight into system performance, and the optimal values of these metrics may also change as iterations progress.

This method is also adaptable to other evolutionary algorithms such as genetic algorithms. For example, although GAs do not have a velocity, the measurement of their population "velocity" diversity can be approached by considering the changes in location of population members from one generation (iteration) to the next in a manner analogous to the velocities in PSO.

This methodology for measuring diversity can facilitate many areas of investigation for PSO and other evolutionary algorithms. Among them are:

1) How should diversity be managed?

2) How does varying parameters and features of PSO such as using the global versus local model, adjusting V_{max}, c_1, c_2, etc. (see Equations 1 and 2) affect diversity?

3) What is the best way to increase or decrease diversity by a predictable quantity?

4) How should each of the three metrics be weighted?

5) Which (if any) statistical attributes of the three metrics are significant and useful?

V. Conclusion

It is important to note that optimal diversity varies with the problem and over the course of a run of an evolutionary algorithm such as PSO. There has been much discussion in the literature of the importance of population diversity. This paper proposes a method to measure it for particle swarm optimization. The method is adaptable for use with other evolutionary algorithms.

The measurement of diversity is only the first step in the eventual management and optimization of diversity. We have much to learn regarding how to determine optimal diversity values, and how these values vary over the process of a run of an evolutionary algorithm. In order to manage diversity, however, we must be first able to measure it. This paper proposes a method to provide this first and important step.

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